

# Improved Bounds for Cops-and-Robber Pursuit

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**Abstract.** We prove that  $n$  cops can capture (that is, some cop can get less than unit distance from) a robber in a continuous square region with side length less than  $\sqrt{5}n$  and hence that  $\lfloor n/\sqrt{5} \rfloor + 1$  cops can capture a robber in a square with side length  $n$ . We extend these results to three dimensions, proving that  $0.34869 \dots n^2 + O(n)$  cops can capture a robber in a  $n \times n \times n$  cube and that a robber can forever evade fewer than  $0.02168 \dots n^2 + O(n)$  cops in that cube.

**Key words.** Pursuit, evasion, cops and robber, lion and man, rabbit and robot

**AMS(MOS) subject classifications.** 49N75

Under what conditions can a robber evade cops on fixed patrol routes (that is, the cops move non-adaptively, independent of the robber's movements)? Pursuit problems have been studied for centuries, with recent results prompted by Dumitrescu, Suzuki, and Zylinski [6] who asked, among other questions, what is the maximum number of cops that a robber can evade, that is, stay at least unit distance away from, on an  $n \times n$  continuous square region; they proved that  $\Omega(\sqrt{n})$  cops can be evaded. Brass, Kim, Na, and Shin [5] improved this by proving that a robber can evade  $\lfloor n/(9\pi + 6) \rfloor = \lfloor n/34.274 \dots \rfloor$  cops. In [1] we further improved that bound by using the results of Berger, Grüne and Klein [3] (who also gave some results for higher dimensions) together with a new discretization lemma to prove that the robber can evade at least  $\lfloor n/5.889 \rfloor$  cops in an  $n \times n$  continuous square region. We also proved in [1] that a robber can always evade a single cop in a square of side length 4, and that a single cop can always capture the robber in a square of side length smaller than  $2.189 \dots$ . Altshuler and Bruckstein [2] examined similar questions for arbitrary connected two-dimensional regions.

In this note we improve on the upper bounds, showing that  $n$  cops moving non-adaptively can capture a robber in a continuous square region with side length less than  $\sqrt{5}n$  and hence that a single cop can capture a robber in a square with side length less than  $\sqrt{5}$ , and hence also that  $\lfloor n/\sqrt{5} \rfloor + 1$  cops can

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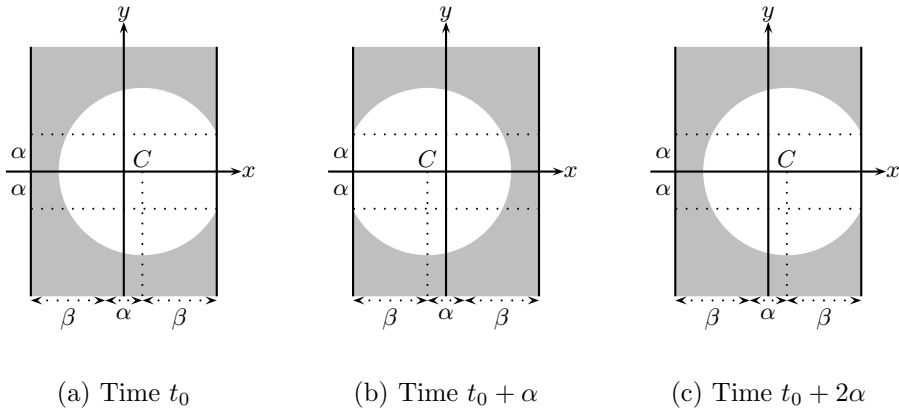


Figure 1: The cop's region of capture (an open disk of radius 1 centered at the cop, shown in white) as he moves from  $(\alpha/2, 0)$  to  $(-\alpha/2, 0)$  and back to  $(\alpha/2, 0)$  at times  $t_0$ ,  $t_0 + \alpha$ , and  $t_0 + 2\alpha$ , respectively, in a strip of width  $\alpha + 2\beta$ .

capture a robber in a square with side length  $n$ . We then extend these results to three dimensions, proving that  $0.34869 \cdots n^2 + O(n)$  cops can capture a robber in a  $n \times n \times n$  cube; we point out a weak lower bound for three dimensions—that a robber can forever evade fewer than  $0.02168 \cdots n^2 + O(n)$  cops in the  $n \times n \times n$  cube.

## 1 Two-Dimensional Upper Bounds

**Theorem 1**  $n$  cops can capture a robber on rectangular strip of width  $w < \sqrt{5}n$ .

*Proof.* Choose two real numbers  $\alpha$  and  $\beta$  such that  $\alpha^2 + \beta^2 < 1$ . Figure 1 shows that in a strip of width  $\alpha + 2\beta$ , one cop moving left and right can prevent a robber from crossing the line  $y = 0$ : if the robber ever has a zero  $y$ -coordinate, the cop will capture the robber, provided that the cop's region of capture<sup>1</sup> at time  $t_0$  includes the point  $(\beta + \alpha/2, \alpha)$ , that is, provided that  $\alpha^2 + \beta^2 < 1$ .

Similarly,  $n$  cops moving synchronously can prevent the robber from crossing a horizontal line in a strip of width  $(\alpha + 2\beta)n$  (see Figure 2). The condition  $\alpha^2 + \beta^2 < 1$  means that the robber cannot pass through the lenticular region of overlap as it opens and closes between adjacent cops.

Thus a chain of oscillating cops can separate the regions above and below the line of the cops. By slightly decreasing the open time of each gap, we can move that chain slowly upward without giving the robber a chance to cross from one region into the other. Indeed, suppose that  $\alpha^2 + \beta^2 \leq 1$ , and choose  $\epsilon$  such that  $0 < \epsilon < \alpha/2$ . A cop can then move upward between the points  $(\frac{\alpha}{2} - \epsilon, 0)$ ,  $(-\frac{\alpha}{2} + \epsilon, \epsilon)$ ,  $(\frac{\alpha}{2} - \epsilon, 2\epsilon)$ ,  $\dots$  and capture a robber on a strip of width  $\alpha + 2\beta - 2\epsilon$ ;

<sup>1</sup>The figures are shown with  $\alpha^2 + \beta^2 = 0.9999$  so the robber can move freely in either gray area,  $y > 0$  or  $y < 0$ , but cannot cross from one to the other without being captured.

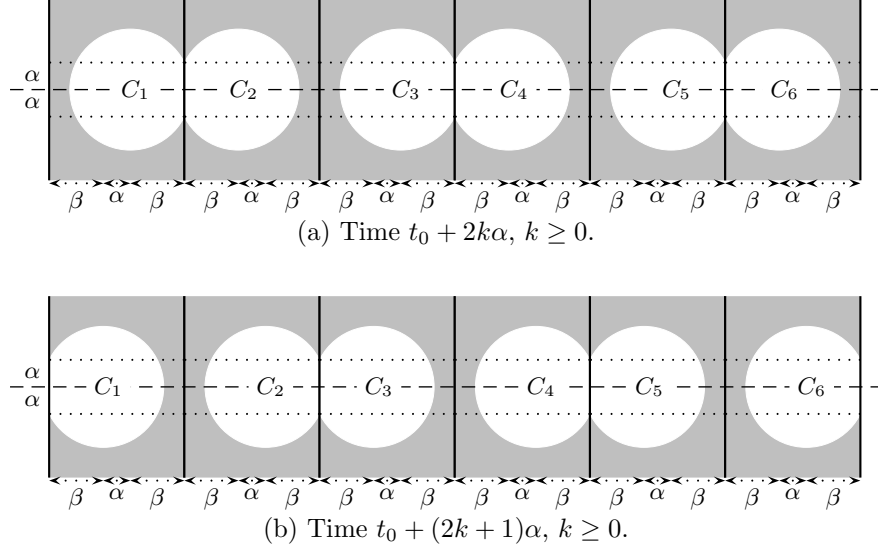


Figure 2:  $n = 6$  cops moving synchronously back and forth on a strip of width of width  $(\alpha + 2\beta)n$ ,  $\alpha^2 + \beta^2 < 1$ , preventing a robber from crossing the horizontal dashed line on which the  $n$  cops move.

thus in each step the region in which the robber can move shrinks in height by at least  $\epsilon$ . Similarly,  $n$  cops moving synchronously can capture a robber on a strip of width  $(\alpha + 2\beta - 2\epsilon)n$ . Maximizing  $\alpha + 2\beta$  subject to  $\beta^2 + \alpha^2 \leq 1$  gives  $\alpha = 1/\sqrt{5}$  and  $\beta = 2/\sqrt{5}$  and so  $\alpha + 2\beta = \sqrt{5}$ .

Given  $n$  cops on a strip of width  $w < \sqrt{5}n$ , divide the rectangular strip of width  $w$  into  $n$  parallel strips of width  $w/n < \sqrt{5}$ . If  $w/n < 2$ , place a cop in the center of the bottom of each strip and have them move upward synchronously, forcing the robber higher and higher in the rectangular strip (this is the trivial algorithm of [1, Sec. 3]). If  $2 \leq w/n < \sqrt{5}$ , choose  $\epsilon = (\sqrt{5} - w/n)/2$ ; then  $0 < \epsilon < \alpha/2$ , so the synchronous upward zig-zagging of the  $n$  cops forces the robber higher and higher in the rectangular strip. In either case, the cops capture the robber at the top of a strip. Assuming the cops move at unit speed in a strip of height  $h$ , elementary computation gives the time until capture as  $h$  for  $w/n < 2$  and as  $h\sqrt{(\alpha - 2\epsilon)^2 + \epsilon^2}/\epsilon = \Theta(h\alpha/\epsilon)$  for  $2 \leq w/n < \sqrt{5}$ .  $\square$

This theorem has two immediate corollaries:

**Corollary 1**  $n$  cops can capture a robber in a square with side length  $\ell < \sqrt{5}n = 2.236 \dots n$ .

**Corollary 2**  $\lfloor n/\sqrt{5} \rfloor + 1$  cops can capture a robber in a square with side length  $n$ .

Because the capture time tends to infinity as  $\ell/n \rightarrow \sqrt{5}$ , we should ask what happens if we bound the capture time.

**Theorem 2** *If the  $n$  cops and the robber all move at most at unit speed in a rectangle of height  $h$ , then for a given time  $t \geq h$ , if the side length  $\ell$  is bounded by*

$$\ell < \frac{n\sqrt{15h^2 + 16ht + 5t^2}}{2h + t},$$

*the cops can capture the robber within time  $t$ .*

*Proof.* We give only an outline of the proof. To simplify the computation with little loss of precision, suppose the cop goes from a bottom point  $(x_0, 0)$  to a top point  $(x_1, h)$  by moving upward along the zig-zag path  $(\alpha/2, 0)$ ,  $(-\alpha/2, \epsilon)$ ,  $(\alpha/2, 2\epsilon)$ ,  $\dots$ . In a single zig-zag step of length  $\sqrt{\alpha^2 + \epsilon^2}$ , the cop moves distance  $\epsilon$  upward. Bounding the capture time by  $t$  means we must have

$$\frac{\epsilon}{\sqrt{\alpha^2 + \epsilon^2}} \geq \frac{h}{t}.$$

To capture the robber at the top, the point  $(\frac{\alpha}{2} + \beta, \sqrt{\alpha^2 + \epsilon^2} + 2\epsilon)$  must be less than distance 1 from the point  $(\alpha/2, 0)$ , giving us a second equation. Maximizing  $\ell = \alpha + 2\beta$  gives us the result for  $n = 1$ ; it extends easily to  $n \geq 1$ .  $\square$

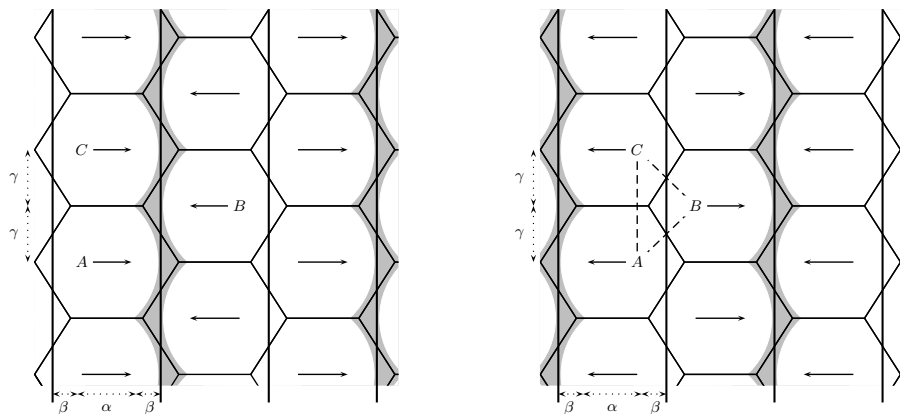
## 2 Three-Dimensional Bounds

To extend the two-dimensional results to three dimensions, imagine a two-dimensional array of cops whose capture radii completely cover an  $n \times n$  square. For example, we can place  $2n^2/(3\sqrt{3}) + O(n) = 0.3849 \dots n^2 + O(n)$  cops in the centers of equilateral regular hexagons with unit side length, tiling the  $n \times n$  square; this honeycomb pattern gives the fewest cops covering the  $n \times n$  square [7] (see also [9]).

If the cops move straight upward (along the  $z$ -axis) synchronously in this pattern, they are guaranteed to capture a robber in an  $n \times n \times n$  cube. We can improve on this trivial result by an analogue of the method in the proof of Theorem 1: have larger hexagons with the cops zig-zagging upward:

**Theorem 3**  $0.34869 \dots n^2 + O(n)$  cops can capture a robber in a  $n \times n \times n$  cube.

*Proof.* We place cops at the centers of hexagons as in the regular honeycomb tiling, but we stretch the hexagons and have the cops zig-zag to and fro in the centers of the hexagons; the vertical stretch factor is  $\gamma$  and the cops' zig-zagging movement is an amount  $\alpha$ , with a horizontal stretch factor  $\beta$ ; this situation is shown in Figure 3. We need to optimize  $\alpha$ ,  $\beta$ , and  $\gamma$ , subject to the constraint that a robber can never cross the plane of the hexagons (that is, the robber will be captured within  $2\alpha$  units of time if he ever enters the plane of the cops'



(a) Time  $t_0 + 2k\alpha$ ,  $k \geq 0$ .

(b) Time  $t_0 + (2k + 1)\alpha$ ,  $k \geq 0$ .

Figure 3: The synchronized to-and-fro movement of hexagonally placed cops in the proof of Theorem 3. Cops  $A$ ,  $B$ , and  $C$ , which form a dashed isosceles triangle in (b), are shown on alternating steps; the squiggly gray strips show the regions not within the capture radius of any cop at the beginning of the step.

movement); as in Theorem 1, we must consider situation when the cops are at the extremes of their movement and verify that the robber cannot get through the squiggly gray strips shown in Figure 3 as the cops zig-zag to and fro.

Consider, for instance, the cops labeled  $A$ ,  $B$ , and  $C$  as shown in Figure 3(b), and let the origin  $(0, 0)$  be at cop  $A$ . The three spheres of capture around the cops are,

$$\begin{aligned} \text{centered at } A: & \quad x^2 + y^2 + z^2 = 1, \\ \text{centered at } B: & \quad (x - 2\beta)^2 + (y - \gamma)^2 + z^2 = 1, \\ \text{centered at } C: & \quad x^2 + (y - 2\gamma)^2 + z^2 = 1. \end{aligned}$$

Ignoring the uninteresting case of  $\beta = \gamma = 0$ , we solve these equations to find the extremes on the  $z$ -axis, the two points where the surfaces of all three spheres intersect. These extremes are  $x = \beta - \gamma^2/(4\beta)$ ,  $y = \gamma$ , and

$$z = \pm \sqrt{1 - \left( \frac{4\beta^2 + \gamma^2}{4\beta} \right)^2}.$$

As in the two-dimensional case, we must have  $2z > 2\alpha$  so that the length of a cop's movement is no greater than the distance a robber would have to traverse to cross the plane of the cops safely in the isosceles triangle between the cops at the end of step 1. Thus we want to maximize  $2\gamma(\alpha + 2\beta)$ , the area controlled by a cop, subject to  $\alpha < z$  (again, we relax the inequality to  $\alpha \leq z$ , compensating

for it later). Elementary calculus gives, with the assistance of Maple, exact algebraic expressions for  $\alpha = 0.29706368521\dots$ ,  $\beta = 0.69303854262\dots$ , and  $\gamma = 0.85194071476\dots$ . Because  $\gamma^2 + \alpha^2 < 1$ , all points not in the isosceles triangles are always threatened by at least one cop. Thus a robber cannot cross the plane of the cops.

With the cop placement as shown in Figure 3, each cop now covers of an area of  $2\gamma(\alpha + 2\beta)$ , so only

$$\frac{n^2}{2\gamma(\alpha + 2\beta)} + O(n) = 0.34869\dots n^2 + O(n)$$

cops are needed to patrol the  $n \times n$  planar region, preventing a robber from crossing it.

Finally, as in the proof of Theorem 1, we reduce  $\alpha$  and  $\beta$  by infinitesimal amounts so that the cops can inch upward, forcing capture of the robber at the top of the  $n \times n \times n$  cube.  $\square$

We can get a very weak lower bound for the three-dimensional case with a computer program that computes, à la [1], the numbers of boundary points for the various cases needed. If we take  $s = 10$  in [1, Thm. 3], we find by exhaustive search that, in the worst case, a cop can threaten at most 1514 vertices accessible to a robber on the grid  $G_{10n}$ . Similarly, if we take  $s = 100$  and divide each grid cell into  $3 \times 3$  sub-grid cells to track the cops' movement as in [1, Thm. 5]), we find that a cop can threaten at most 115298 vertices accessible to a robber on the grid  $G_{100n}$ . Now, mimicking the proof of [3, Lem. 10], we find that on a grid  $G_n$ , if  $v$  is the number of vertices potentially threatened by a cop, at least

$$\frac{L(\lfloor 3(n-1)/2 \rfloor, n, 3)}{3v} + O(n)$$

cops are needed to clear the grid. From [8, A077043],

$$L(\lfloor 3(n-1)/2 \rfloor, n, 3) = \frac{3}{4}n^2 + O(n),$$

giving a lower bound of

$$\frac{100^2}{4 \times 115298}n^2 + O(n) \approx 0.021683n^2 + O(n).$$

### 3 Conclusions

In [1, Thms. 6 and 8] we proved that the robber can forever evade capture by a single cop on fixed patrol in a square with a side length  $\ell \geq 4$  and that a single cop on fixed patrol can capture a robber in a square with side length  $\ell < 2.189\dots$ ; Corollary 1 with  $n = 1$  improves on the upper bound, without violating the conjecture [1, Conj. 5] that  $\ell < 2.2657548\dots$  is the best possible bound. Corollary 2 improves on the obvious lower bound that  $\lceil n/2 \rceil$  cops can capture

a robber on a square with a side length  $\ell < n$ , and disproves our conjecture [1, Conj. 1] that for  $n \geq 3$ , the robber can forever evade capture if there are at most  $\lfloor n/2 \rfloor$  cops. (Brass [4, Prob. 13], noted implicitly that our conjecture was wrong; he suggested, without analysis, cop movement similar to but simpler than that in our Theorem 1.) Corollary 2 should be compared to [1, Thm. 5] which states that if there are at most  $\lfloor n/5.889 \rfloor$  cops, the robber can forever evade capture in the  $n \times n$  square. For the three-dimensional case, the simple zig-zag movement of Theorem 3, a generalization of the two-dimensional result of Corollary 2, results about in a 10% improvement over the trivial algorithm.

Berger, Grüne and Klein [3] analyzed boundary vertices to prove  $\Theta(n^{d-1}/\sqrt{d})$  cops on fixed patrol are necessary to capture a robber on an  $d$ -dimensional  $n \times \dots \times n$  grid; thus by appropriately discretizing the continuous  $n \times n \times \dots \times n$  region (mimicking the two-dimensional proof in [1]), it follows that for fixed dimension  $d$ ,  $\Theta(n^{d-1})$  such cops are necessary to guarantee capture in the  $d$ -dimensional cube. To get further results, it might be possible to refine the weak lower bound at the the end of Section 2; but it would be very difficult to obtain the isoperimetric lemmas needed to derive purely analytical bounds as [1] did in the two-dimensional case.

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